

Q1. a) Dimensions:

i) $\text{Dim } S = 4$ because it is a two by two matrix.

ii) $\text{Dim } y = 2$ because it is a one by two matrix.

iii) $\text{Dim } \bar{y} = \text{Dimensionless}$ because it is a constant.

iv) $\text{Dim } y_1 = \text{Dimensionless}$ because it is a constant.

$$b) \cdot c = w'y \\ = (w_1, w_2) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = w_1 y_1 + w_2 y_2$$

$$S = \begin{bmatrix} a & p \\ p & b \end{bmatrix}$$

$$E(c) = E(w_1 y_1 + w_2 y_2)$$

$$E(w'y) = E(w_1 y_1) + E(w_2 y_2)$$

$$E(w'y) = w_1 E(y_1) + w_2 E(y_2)$$

$$M_c = w_1 M_{y_1} + w_2 M_{y_2}$$

$a = \text{variance of } y_1$

$b = \text{variance of } y_2$

$p = \text{cov}(y_1, y_2)$

$$\text{Var}(c) = E((c - M_c))^2$$

$$= E((w_1 y_1 + w_2 y_2) - (w_1 M_{y_1} + w_2 M_{y_2}))^2$$

$$= E((w_1 (y_1 - M_{y_1}) + w_2 (y_2 - M_{y_2}))^2)$$

$$= E((w_1 (y_1 - M_{y_1}))^2 + (w_2 (y_2 - M_{y_2}))^2)$$

$$= w_1^2 E[(y_1 - M_{y_1})^2] + 2w_1 w_2 E[(y_1 - M_{y_1})(y_2 - M_{y_2})] + w_2^2 E[(y_2 - M_{y_2})^2]$$

$$= w_1^2 \text{Var}(y_1) + 2w_1 w_2 \text{Cov}(y_1, y_2) + w_2^2 \text{Var}(y_2)$$

$$= w_1^2 a + 2w_1 w_2 p + w_2^2 b$$

$$\text{Var}(c) = a w_1^2 + 2w_1 w_2 p + b w_2^2$$

$$\text{C) } \text{Var } C = a\omega_1^2 + 2\omega_1\omega_2\rho + b\omega_2^2$$

$$\frac{\partial C}{\partial \omega_1} = 2a\omega_1 + 2\omega_2\rho = 0$$

$$2\omega_1$$

$$\frac{\partial C}{\partial \omega_2} = 2\omega_1\rho + 2b\omega_2 = 0$$

$$2\omega_2$$

$$2a\omega_1 = -2\omega_2\rho$$

$$2b\omega_2 = -2\omega_1\rho$$

$$\frac{2a\omega_1}{2b\omega_2} = \frac{-2\omega_2\rho}{-2\omega_1\rho}$$

$$a\omega_1^2 = b\omega_2^2$$

$$a\omega_1^2 - b\omega_2^2 = 0$$

$$\text{Rem } \omega_1^2 + \omega_2^2 = 1 \Rightarrow \omega_1^2 = 1 - \omega_2^2$$

$$a(1 - \omega_2^2) - b\omega_2^2 = 0$$

$$a - a\omega_2^2 - b\omega_2^2 = 0$$

$$\omega_2^2(a+b) = a$$

$$\omega_2^2 = \frac{a}{a+b}$$

$$\Rightarrow \omega_2 = \sqrt{\frac{a}{a+b}}$$

$$\omega_1^2 = 1 - \frac{a}{a+b}$$

$$\Rightarrow \omega_1 = \sqrt{1 - \frac{a}{a+b}}$$

Q(d and e)

d) $a=1, b=1, p=1/2$

$$2w_1 + w_2 = 0$$

$$w_1 + 2w_2 = 0$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation of A is:

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)^2 - 1^2 = 0$$

$$4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-1)(\lambda-3) = 0$$

$$\lambda = 1, 3$$

$$\tilde{\lambda} = 1 \text{ and } \hat{\lambda} = 3$$

$$(A - \lambda I)v = 0$$

$$\lambda = 1$$

$$(A - I)v = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 = 0$$

$$v_1 = -v_2$$

$$\text{let } v_1 = w$$

$$\text{let } v_2 = t$$

$$\tilde{w} = (-t, t) = t(-1, 1)$$

$$\tilde{w} = (-1, 1)$$

$$\text{For } \hat{\lambda} = 3$$

$$(A - 3I)v = 0$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-v_1 + v_2 = 0$$

$$v_1 = v_2 \text{ let } v_2 = t$$

$$\hat{w} = (t, t) = t(1, 1)$$

$$\hat{w} = (1, 1)$$

The eigen vector $\tilde{w}$ satisfies the first principal component. The second eigen vector $\hat{w}$ corresponds to the second principal component since it is orthogonal to $\tilde{w}$.

G.f.

Since $A=25$ we have;

$$\tilde{\lambda} = \frac{1}{2} \text{ and } \hat{\lambda} = \frac{3}{2}$$

$$\tilde{w} = (-1, 1) \text{ and } \hat{w} = (1, 1)$$

The orthonormal vectors are;

$$\vec{\tilde{w}} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \text{ and } \vec{\hat{w}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$S = \tilde{\lambda} \cdot \tilde{w}(\tilde{w})^T + \hat{\lambda} \hat{w}(\hat{w})^T$$

$$= \frac{1}{2} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \frac{3}{2} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$